第五届 Xionger 网络数学竞赛试卷

(代数与数论组, 2022 年 9 月 10 日至 9 月 12 日)

考试时间：2022 年 9 月 10 日上午 9 点至 9 月 12 日晚上 21 点
官方微信公众号：**Xionger 的数学小屋**

每题暂不设分值，希望诸位尽可能多地作答。试题解答请及时发送到邮箱 2609480070@qq.com, 逾期将取消参赛资格. 要求解答字迹清楚, 排版美观, 推荐采用 PDF 文档格式提交, 文件命名: **代数与数论组 + 昵称（或姓名）+ 学校**.

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1. Let $(A, \mathfrak{m})$ be a local Noetherian ring (with element 1) and M be a finitely generated $A-$ module. A free resolution of M is an exact sequence

$$\cdots \xrightarrow{\varphi_{k+2}} F_{k+1} \xrightarrow{\varphi_{k+1}} F_k \xrightarrow{\varphi_k} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

with finitely generated free $A-$ modules F_i for $i \geq 0$.

(a) Consider $M/\mathfrak{m}M$ as a $A/\mathfrak{m}-$ vector space. Let $\{x_1, \ldots, x_n\}$ be a minimal set of generators of M. Prove that $\{\overline{x_1}, \ldots, \overline{x_n}\}$ is a basis of the vector space $M/\mathfrak{m}M$, where $\overline{x_i} = x_i + \mathfrak{m}M \in M/\mathfrak{m}M$.

(b) A free resolution as above is called minimal free resolution if $\varphi_k(F_k) \subseteq \mathfrak{m}F_{k-1}$ for $k \geq 1$. Prove that M has a minimal free resolution.

(c) If M has two minimal free resolutions

$$\cdots \xrightarrow{\varphi_{k+2}} F_{k+1} \xrightarrow{\varphi_{k+1}} F_k \xrightarrow{\varphi_k} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0,$$

$$\cdots \xrightarrow{\psi_{k+2}} G_{k+1} \xrightarrow{\psi_{k+1}} G_k \xrightarrow{\psi_k} \cdots \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0 \xrightarrow{\psi_0} M \to 0,$$

prove that $\operatorname{rank}(F_k) = \operatorname{rank}(G_k)$ for $k \geq 0$.

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2. k is a field. Let E be an algebraic extension of k, and let $\sigma : E \to E$ be an embedding of E into itself over k. Then σ is an automorphism.

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3. k is a field. Let V be a $k-$ vector space. W is a subspace of V, $T : V \to V$ is a injective $k-$ linear map such that $T(W) \subseteq W$. Suppose V/W and $W/T(W)$ are finite-dimensional $k-$ vector spaces. Prove that as $k-$ vector spaces, $T(V)/(W \cap T(V))$ and $(W + T(V))/W$ have same dimension. $W/(W \cap T(V))$ and $(W + T(V))/T(V)$ have same dimension. $V/(W + T(V))$ and $(W \cap T(V))/T(W)$ have same dimension. $V/T(V)$ and $W/T(W)$ have same dimension.

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